Application of He's Homotopy Perturbation Method for Solving Fractional Fokker-Planck Equations

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The fractional Fokker-Planck equation (FFPE) has been used in many physical transport problems which take place under the influence of an external force field and other important applications in various areas of engineering and physics. In this paper, by means of the homotopy perturbation method (HPM), exact and approximate solutions are obtained for two classes of the FFPE initial value problems. The method gives an analytic solution in the form of a convergent series with easily computed components. The obtained results show that the HPM is easy to implement, accurate and reliable for solving FFPEs. The method introduces a promising tool for solving other types of differential equation with fractional order derivatives.

Key words: Fractional Fokker-Planck Equation; Homotopy Perturbation Method; Riemann-Liouville Fractional Derivative.

1. Introduction

The Fokker-Planck equation (named after Adriaan Fokker and Max Planck) describes the time evolution of the probability density function of position and velocity of a particle, which is one of the classical, widely used equations of statistical physics. In the presence of an external force field $F(x) = -U'(x)$, the evolution of a test particle is usually described in terms of the Fokker-Planck equation (FPE) [1]:

$$\frac{\partial W}{\partial t} = \left[\frac{\partial}{\partial x}\frac{U'(x)}{m\eta_1} + k_1\frac{\partial^2}{\partial x^2}\right] W(x,t), \qquad (1)$$

where $W(x,t)$ defines the probability of finding the particle at a certain position x at a given time t, m denotes the mass of the particle, k_1 the diffusion constant associated with the transport process, and the friction coefficient η_1 is a measure for the interaction of the particle with its environment. Some properties of the FPE (1) are mentioned by Metzler et al. [2]. These properties cause that the conventional FPE (1) is not adequate to describe anomalous subdiffusion. As a model for characterizing subdiffusion, in an external potential field $U(x)$, the fractional Fokker-Planck equation (FFPE)

$$\frac{\partial W}{\partial t} = {}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\frac{U'(x)}{m\eta_\alpha} + k_\alpha\frac{\partial^2}{\partial x^2}\right] W(x,t), \qquad (2)$$

has been suggested [3 – 6]. Here, $W(x,t)$ is the probability density, $U(x)$ indicates the potential of overdamped Brownian motion, a prime stands for the derivative with respect to the space coordinate, k_α denotes the generalized diffusion coefficient with physical dimension [$\mathrm{cm}^2\,\mathrm{s}^{-\alpha}$], and η_α is the generalized friction coefficient possessing the dimension [$\mathrm{s}^{\alpha-2}$] and ${}_0D_t^{1-\alpha}$ stands for the Riemann-Liouville fractional derivative of order $1-\alpha$, which is defined as follows [7 – 9]:

$${}_0D_t^{1-\alpha}W(x,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial'}{\partial t}\int_0^t \frac{W(x,s)}{(t-s)^{1-\alpha}}\mathrm{d}s, \qquad (3)$$

where $\alpha \in [0,1)$, and $\Gamma(x)$ is the gamma function. The fundamental property of the Riemann-Liouville fractional derivative is the fractional differentiation of a power,

$${}_0D_t^{1-\alpha}t^q = \frac{\Gamma(q+1)}{\Gamma(q+\alpha)}t^{q+\alpha-1}, \quad \text{for any real } q. \qquad (4)$$

It is well-known that among many other applications, the model of a Brownian particle in a periodic potential can be used to describe Brownian motors and molecular motors [10–12], such as kinesins, dyneins, and myosins. However, many other systems, such as RNA polymerases, exonuclease and DNA polymerases, helicases, the motion of ribosomes along mRNA, and the translocation of RNA or DNA through a pore, are advantageously described as particles moving along a disordered substrate. Depending on the statistical properties of the potential, the long-time limit of the process can be quite different from that in a washboard potential [13]. It has been shown that the heterogeneity of the substrate potential may lead to anomalous dynamics [14, 15]. In particular, over a range of forces around the stall force subdiffusion is observed [24]; i. e. the displacement grows as $\langle \delta r^2(t) \rangle \sim t^{\alpha}$ with $0 < \alpha < 1$.

It should be noted that the FFPE (2) with fractional order of time derivation happens in discontinuous time in large time scales in weather forecast or in the very small time scales in high energy physics. In this problem the discontinuous time reflects almost periodic discontinuity of time. Discontinuing of time happens according to the E-infinity theory, and the fractional model is the best candidate to describe such problems. Time-fractional equations always behave fascinatingly as illustrated in [16, 17].

There is a more general form of the Fokker-Planck equation. Nonlinear FPE has important applications in various areas such as plasma physics, surface physics, population dynamics, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology, and marketing (see [18, 19] and references therein). For N variables $x_1, \dots, x_N$, the general nonlinear FFPE can be written in the following form:

$$\frac{\partial W}{\partial t} = {}_0D_t^{1-\alpha}\Bigg[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(\mathbf{x},t,W) + \sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(\mathbf{x},t,W)\Bigg]W(\mathbf{x},t), \tag{5}$$

where $\mathbf{x} = (x_1, \dots, x_N)$, A_1 and $B_{i,j}$ are known functions.

Because of the large number of applications of the Fokker-Planck equation, a lot of work is done in order to find the numerical solution of this equation. For examples, we refer the readers to [2, 5, 6, 19, 20].

Fractional differential equations have been caught much attention recently due to the exact description of nonlinear phenomena. It is known that no analytical method was available before 1998 for such equations even for linear fractional differential equations. In 1998, the variational iteration method (VIM) was first proposed to solve fractional differential equations with greatest success, see [21]. From then the variational iteration method and the homotopy perturbation method (HPM) became the best candidates for solving various fractional equations. For example, the variational iteration method is employed in [22] for solving fractional vibration equations and in [23] for solving a Fokker-Planck equation. Application of the VIM and HPM to the fractional evolution equations is investigated in [24]. He's homotopy perturbation method is used for solving nonlinear partial differential equations of fractional order [25]. In [26], the applications of the HPM method to coupled systems of partial differential equations with time fractional derivatives are provided.

In the present work, we construct solutions for the two classes of FFPE, (2) and (5), using He's HPM. In recent years a lot of attention has been drawn to study the homotopy perturbation method to investigate various scientific models. The HPM, based on series approximation, is one among the newly developed analytical methods for strongly nonlinear problems and has been proven successful in solving a wide class of differential equations [24–37]. The method provides the solution in a rapidly convergent series with components that can be simply computed. The HPM is useful for obtaining both closed form explicit solutions and numerical approximations of linear or nonlinear differential equations and it is of great interest to applied science, engineering, physics, biology, etc.

2. Basic Idea of the Homotopy Perturbation Method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation [27]:

$$L(W) + N(W) - f(r) = 0, \quad r \in \Omega, \tag{6}$$

with the boundary conditions

$$B(W, \partial W/\partial n) = 0, \quad r \in \Pi, \tag{7}$$

where L and N are linear and nonlinear differential operators, respectively, B a boundary operator, $f(r)$ a known analytical function and Π is the boundary of the domain Ω.

By the homotopy technique, we construct a homotopy $V(r,p):\Omega\cdot[0,1]\to\mathbb{R}$ which satisfies

$$\begin{aligned}H(V,p)&=(1-p)\,[L(V)-L(W_0)]\\&\quad+p\,[L(V)+N(V)-f(r)]=0,\qquad(8)\\p&\in[0,1],\ r\in\Omega,\end{aligned}$$

where $p\in[0,1]$ is an embedding parameter, W_0 is an initial approximation of (6), which satisfies the boundary conditions (7).

Obviously, from (8) we will have

$$H(V,0)=L(V)-L(W_0)=0, \tag{9}$$

$$H(V,1)=L(V)+N(V)-f(r)=0, \tag{10}$$

the changing process of p from zero to unity is just that of $V(r,p)$ from $W_0(r)$ to $W(r)$.

According to the HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solution of (8) can be written as a power series in p:

$$V=V_0+pV_1+p^2V_2+\ldots. \tag{11}$$

Setting $p=1$ results in the approximate solution of (6):

$$W=\lim_{p\to1}V=V_0+V_1+V_2+\ldots. \tag{12}$$

The series in (12) is convergent for most cases, and also the rate of convergence depends on the linear and nonlinear differential operators [27].

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method, which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have the full advantage of the traditional perturbation techniques.

3. Applications

3.1. The FFPE Used for Characterizing Subdiffusion

Consider the FFPE initial value problem

$$\begin{aligned}&\frac{\partial W}{\partial t}={}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\frac{U'(x)}{m\eta_\alpha}+k_\alpha\frac{\partial^2}{\partial x^2}\right]W(x,t),\\&W(x,0)=\varphi(x).\end{aligned}\tag{13}$$

According to (8), a homotopy $V(x,t,p):\Omega\cdot[0,1]\to\mathbb{R}$ can be constructed as follows:

$$\begin{aligned}&(1-p)\left(\frac{\partial V}{\partial t}-\frac{\partial W_0}{\partial t}\right)\\&+p\left(\frac{\partial V}{\partial t}-{}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\frac{U'(x)}{m\eta_\alpha}+k_\alpha\frac{\partial^2}{\partial x^2}\right]V\right)=0,\\&p\in[0,1],\quad(x,t)\in\Omega,\end{aligned}\tag{14}$$

where $W_0=V_0(x,0)=W(x,0)$.

One can now try to obtain a solution of (14) in the form of

$$V(x,t)=V_0(x,t)+pV_1(x,t)+p^2V_2(x,t)+\ldots. \tag{15}$$

Substituting (15) into (14), and arranging the coefficients of "p" powers, yields:

$$\begin{aligned}&p^0:\frac{\partial V_0}{\partial t}=0,\\&p^1:\frac{\partial V_1}{\partial t}={}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\frac{U'(x)}{m\eta_\alpha}+k_\alpha\frac{\partial^2}{\partial x^2}\right]V_0,\\&p^2:\frac{\partial V_2}{\partial t}={}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\frac{U'(x)}{m\eta_\alpha}+k_\alpha\frac{\partial^2}{\partial x^2}\right]V_1,\\&\vdots\\&p^n:\frac{\partial V_n}{\partial t}={}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\frac{U'(x)}{m\eta_\alpha}+k_\alpha\frac{\partial^2}{\partial x^2}\right]V_{n-1},\\&n=3,4,5,\ldots,\end{aligned}\tag{16}$$

with the following initial conditions:

$$V_i(x,0)=\begin{cases}\varphi(x),\ i=0,\\0,\qquad i=1,2,3,\ldots\end{cases}\tag{17}$$

Example 1. Firstly, we consider the FFPE (13) with

$$\frac{U'(x)}{m\eta_\alpha}=-1,\ k_\alpha=1,\ \text{and}\ \varphi(x)=x(1-x),\tag{18}$$

which is considered by Chen et al. [6].

Using above parameter values and (4), the solution of the system (16), according to conditions (17), can be easily obtained by integrating each equation of the system with respect to t as follows:

$$\begin{aligned}&V_0(x,t)=x(1-x),\\&V_1(x,t)=\frac{(2x-3)t^\alpha}{\alpha\Gamma(\alpha)},\\&V_2(x,t)=-\frac{2t^{2\alpha}}{2\alpha\Gamma(2\alpha)},\\&V_n(x,t)=0,\quad\text{for } n=3,5,6,\ldots\end{aligned}\tag{19}$$

Substituting (19) into (12) yields

$$W(x,t)=x(1-x)+\frac{(2x-3)t^{\alpha}}{\Gamma(\alpha+1)}-\frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}, \quad (20)$$

which is the exact solution of the FFPE (13) with the parameter values (18).

Example 2. Secondly, consider the FFPE (13) with

$$\frac{U'(x)}{m\eta_{\alpha}}=\frac{1}{x+1},\ k_{\alpha}=1,\ \text{and}\ \varphi(x)=(x+1)^3, \quad (21)$$

which is considered by Chen et al. [6].

Using parameter values in (21) and (4), we can simply obtain the solution of the system (16), according to conditions (17) as follows:

$$V_0(x,t)=(x+1)^3, \quad V_1(x,t)=\frac{8(x+1)t^{\alpha}}{\alpha\Gamma(\alpha)}, \quad (22)$$
$$V_n(x,t)=0, \quad \text{for } n=2,4,5,\ldots.$$

Substituting (22) into (12) yields the exact solution of the FFPE (13) with the parameter values (21),

$$W(x,t)=(x+1)^3+\frac{8(x+1)t^{\alpha}}{\Gamma(\alpha+1)}. \quad (23)$$

Example 3. Consider the FFPE (13) with

$$U(x)=\cos(x)-6x,\ m\eta_{\alpha}=6,\ k_{\alpha}=2, \text{ and } \varphi(x)=0.1, \quad (24)$$

which is considered by Deng [5].

By substituting (24) into system (16) with initial conditions (17), we continued solving (16) for V_n, $n=0,1,\ldots$ until V_5 and hence obtain a six-term approximation $\psi_5=\sum_{n=0}^{5}V_n$. The components of ψ_5 can be easily obtained with the aid of the relation (4). The closed form approximate solution of (13) with the parameter values of (24) using HPM is given by

$$\begin{aligned}
\psi_5(x,t)&=\frac{1}{10}-\frac{t^{\alpha}}{60\Gamma(\alpha+1)}\cos(x)\\
&+\frac{t^{2\alpha}}{360\Gamma(2\alpha+1)}\big[2\cos^2(x)-6\sin(x)+12\cos(x)-1\big]\\
&+\frac{t^{3\alpha}}{2160\Gamma(3\alpha+1)}\big[-6\cos^3(x)-120\cos^2(x)\\
&+36\sin(x)\cos(x)+144\sin(x)-103\cos(x)+60\big]\\
&+\frac{t^{4\alpha}}{12960\Gamma(4\alpha+1)}\big[24\cos^4(x)+1008\cos^3(x)\\
&-216\cos^2(x)\sin(x)-3456\sin(x)\cos(x)\\
&+5516\cos^2(x)-2310\sin(x)-360\cos(x)-2767\big]\\
&-\frac{t^{5\alpha}}{77760\Gamma(5\alpha+1)}\big[120\cos^5(x)+8640\cos^4(x)\\
&-1440\cos^3(x)\sin(x)-51840\cos^2(x)\sin(x)\\
&+121428\cos^3-236268\sin(x)\cos(x)\\
&+216096\cos^2(x)-16920\sin(x)-101963\cos(x)\\
&-111288\big]. \quad (25)
\end{aligned}$$

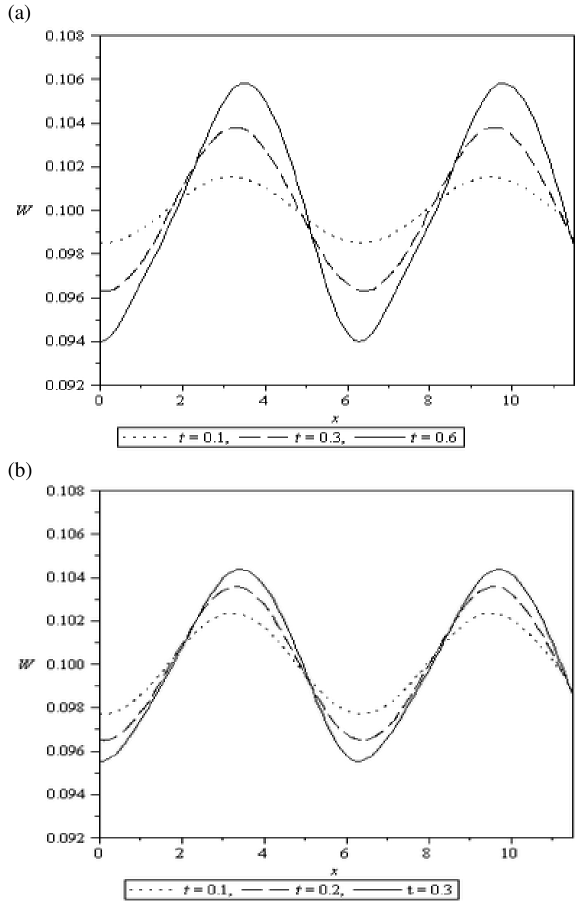


Fig. 1. Behaviour of the probability density $W(x,t)$, estimated by ψ_5, versus x for different values of t with $\alpha=1.0$, shown in (a), and $\alpha=0.8$, shown in (b), for the FFBE (13) with the parameter values of (24).

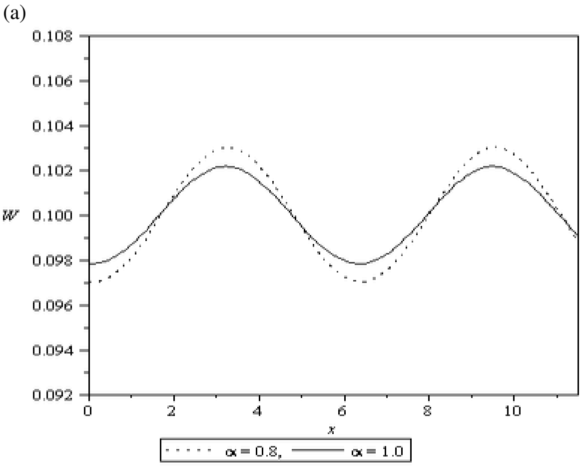
The behaviour of the probability density $W(x,t)\simeq\psi_5$ versus x, for different values of t and α, is shown in Figures 1 and 2.

3.2. The General FFPE

Consider the general FFPE initial value problem

$$\frac{\partial W}{\partial t}={}_0D_t^{1-\alpha}\Big[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(\mathbf{x},t,W)$$

(a)



(b)

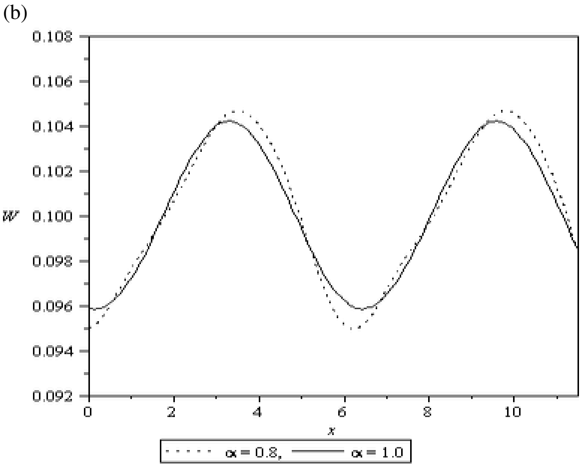


Fig. 2. Behaviour of the probability density $W(x,t)$, estimated by ψ_5, versus x for different values of α at $t = 0.15$, shown in (a), and $t = 0.35$, shown in (b), for the FFBE (13) with the parameter values of (24).

$$+\sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(\mathbf{x},t,W)\Bigg]W(\mathbf{x},t),$$
$$W(\mathbf{x},0)=\varphi(\mathbf{x}). \tag{26}$$

According to (8), a homotopy $V(\mathbf{x},t,p):\Omega\cdot[0,1]\to\mathbb{R}$ can be constructed as follows:

$$\begin{aligned}&(1-p)\left(\frac{\partial V}{\partial t}-\frac{\partial W_0}{\partial t}\right)\\&+p\left(\frac{\partial V}{\partial t}-{}_0D_t^{1-\alpha}\left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_1}A_1(\mathbf{x},t,V)\right.\right.\\&\left.\left.\qquad+\sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i\partial x_j}B_{i,j}(\mathbf{x},t,V)\right]V\right)=0,\end{aligned} \tag{27}$$

where $W_0 = V_0(\mathbf{x},0) = W(\mathbf{x},0)$.

One can now try to obtain a solution of (27) in the form of

$$V(\mathbf{x},t)=V_0(\mathbf{x},t)+pV_1(\mathbf{x},t)+p^2V_2(\mathbf{x},t)+\dots. \tag{28}$$

Substituting (28) into (27), and arranging the coefficients of "p" powers, yields a system of differential equations that depend on functions A_i and $B_{i,j}$ with the following initial conditions:

$$V_i(\mathbf{x},0)=\begin{cases}\varphi(\mathbf{x}), & i=0,\\ 0, & i=1,2,3,\dots.\end{cases} \tag{29}$$

Example 4. Consider the FFPE (26) with $N = 2$, $\mathbf{x} = (x,y)$, and

$$\begin{aligned}&A_1(x,y)=x,\quad A_2(x,y)=5y,\\&B_{1,1}(x,y)=x^2,\quad B_{1,2}(x,y)=1,\\&B_{2,1}(x,y)=1,\quad B_{2,2}(x,y)=y^2,\text{ and}\\&\varphi(x,y)=x,\quad x,y\in\mathbb{R},\end{aligned} \tag{30}$$

which is considered by Tatari et al. [19] in case of $\alpha = 1$.

Substituting (28) into (27) with the parameter functions of (30), and arranging the coefficients of "p" powers, yields

$$\begin{aligned}&p^0: V_{0_t}=0,\\&p^1: V_{1_t}={}_0D_t^{1-\alpha}[x^2V_{0_{xx}}+y^2V_{0_{yy}}\\&\qquad+2V_{0_{xy}}+3xV_{0_x}-yV_{0_y}-2V_0],\\&p^2: V_{2_t}={}_0D_t^{1-\alpha}[x^2V_{1_{xx}}+y^2V_{1_{yy}}\\&\qquad+2V_{1_{xy}}+3xV_{1_x}-yV_{1_y}-2V_1],\\&\vdots\\&p^n: V_{n_t}={}_0D_t^{1-\alpha}[x^2V_{n-1_{xx}}+y^2V_{n-1_{yy}}\\&\qquad+2V_{n-1_{xy}}+3xV_{n-1_x}-yV_{n-1_y}-2V_{n-1}],\\&n=3,4,5,\dots,\end{aligned} \tag{31}$$

with the following initial conditions:

$$V_i(x,y,0)=\begin{cases}x, & i=0,\\ 0, & i=1,2,3,\dots.\end{cases} \tag{32}$$

With the aid of (4), solutions of some equations of the system (31) with respect to initial conditions (32) give the first few components of a series solution $V(x,y,t)$ that are derived as follows:

$$V_0(x,y,t)=x,\quad V_1(x,y,t)=x\frac{t^\alpha}{\alpha\Gamma(\alpha)},$$
$$V_2(x,y,t)=x\frac{t^{2\alpha}}{2\alpha\Gamma(2\alpha)},\quad V_3(x,y,t)=x\frac{t^{3\alpha}}{3\alpha\Gamma(3\alpha)},$$

$$V_4(x,y,t) = x\frac{t^{4\alpha}}{4\alpha\Gamma(4\alpha)}, \quad V_5(x,y,t) = x\frac{t^{5\alpha}}{5\alpha\Gamma(5\alpha)}, \quad \ldots . \tag{33}$$

Therefore:

$$\begin{aligned} W(x,y,t) = x\Big(1 &+ \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\ &+ \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \ldots\Big). \end{aligned} \tag{34}$$

So the solution of the FFPE (26), with $N = 2$ and functions of (30), in a closed form is

$$W(x,y,t) = x\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{35}$$

In case of $\alpha = 1$, the exact solution of the problem considered by Tatari et al. [19] is directly obtained:

$$W(x,y,t) = x\exp(t). \tag{36}$$

Example 5. Finally, we applied the HPM to nonlinear FFPE. Consider the FFPE (26) with $N = 1$, $\mathbf{x} = x$, and

$$\begin{aligned} &A_1(x,t,W) = \frac{4W}{x} - \frac{x}{3}, \quad B_{1,1}(x,t,W) = W, \\ &\text{and } \varphi(x) = x^2, \quad x \in \mathbb{R}, \end{aligned} \tag{37}$$

which is considered by Tatari et al. [19] in case of $\alpha = 1$.

Substituting (28) into (27) with the parameter functions of (37), and arranging the coefficients of "p" powers, yields

$$\begin{aligned} p^0 :\ & V_{0_t} = 0, \\ p^1 :\ & V_{1_t} = {}_0D_t^{1-\alpha}\Big[\Big(2V_{0_{xx}} - \frac{8}{x}V_{0_x} + \frac{4}{x^2}V_0 + \frac{1}{3}\Big)V_0 \\ & + \Big(2V_{0_x} + \frac{x}{3}\Big)V_{0_x}\Big], \\ p^2 :\ & V_{2_t} = {}_0D_t^{1-\alpha}\Big[\Big(2V_{0_{xx}} - \frac{8}{x}V_{0_x} + \frac{4}{x^2}V_0 + \frac{1}{3}\Big)V_1 \\ & + \Big(4V_{0_x} + \frac{x}{3}\Big)V_{1_x} + \Big(2V_{1_{xx}} - \frac{8}{x}V_{1_x} + \frac{4}{x^2}V_1\Big)V_0\Big], \\ p^3 :\ & V_{3_t} = {}_0D_t^{1-\alpha}\Big[\Big(2V_{0_{xx}} - \frac{8}{x}V_{0_x} + \frac{4}{x^2}V_0 + \frac{1}{3}\Big)V_2 \\ & + \Big(4V_{0_x} + \frac{x}{3}\Big)V_{2_x} + \Big(2V_{1_{xx}} - \frac{8}{x}V_{1_x} + \frac{4}{x^2}V_1\Big)V_1 \\ & + 2V_{1_x}V_{1_x} + \Big(2V_{2_{xx}} - \frac{8}{x}V_{2_x} + \frac{4}{x^2}V_2\Big)V_0\Big], \\ \vdots \end{aligned} \tag{38}$$

with the initial conditions

$$V_i(x,0) = \begin{cases} x^2, \, i = 0, \\ 0, \ \ i = 1,2,3,\ldots \end{cases}. \tag{39}$$

Now, solving the system (38) according to conditions (39), we obtain

$$\begin{aligned} &V_0(x,t) = x^2, \ V_1(x,t) = x^2\frac{t^{\alpha}}{\alpha\Gamma(\alpha)}, \\ &V_2(x,t) = x^2\frac{t^{2\alpha}}{2\alpha\Gamma(2\alpha)}, \ V_3(x,t) = x^2\frac{t^{3\alpha}}{3\alpha\Gamma(3\alpha)}, \\ &\ldots \end{aligned} \tag{40}$$

Hence, the first 4-term HPM series solution can be written as

$$\begin{aligned} W(x,t) = x^2\Big(1 &+ \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\Big). \end{aligned} \tag{41}$$

So the solution of the FFPE (26), with $N = 1$ and functions of (37), can be written in a closed form as

$$W(x,t) = x^2\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{42}$$

In case of $\alpha = 1$, the exact solution of the problem considered by Tatari et al. [19] is directly obtained:

$$W(x,t) = x^2\exp(t). \tag{43}$$

4. Conclusions

The homotopy perturbation method was implemented successfully for solving two classes of fractional Fokker-Planck equation initial value problems. It can be concluded that the HPM is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of fractional linear and nonlinear partial differential equations. It provides the solution in terms of convergent series with easily computable components in a straightforward manner without using restrictive assumptions or linearization. It is important to note that the HPM method provides a closed form of the solution that rapidly converges to the exact one if it's free of singularities. Finally, the study shows that the HPM requires less computational work than existing approaches while supplying quantitatively reliable results.

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